Derivation of a formula for the implicit derivative

Suppose $g(x, y)$ is a function of two variables and c is a constant. Then $g(x, y) = c$ determines a level set A at level c in the xy-coordinate plane. Choose a point (p, q) in A. We know that if A is a curve, then the slope of the tangent to A at (p, q) gives us the implicit derivative which is naturallly denoted $\frac{dy}{dx}$. (Draw a picture.)

We wish to calculate the implicit derivative $\frac{dy}{dx}$ of g at the point (p, q).

Suppose $(p + \Delta x, q + \Delta y)$ is another point in A close to (p, q). Then the change in g between the two given points in A is given by

$$\begin{aligned}\Delta g \quad &= g(p + \Delta x, q + \Delta y) - g(p, q)\\ &= g(p + \Delta x, q + \Delta y) - g(p + \Delta x, q) + g(p + \Delta x, q) - g(p, q).\end{aligned}$$

(The first equation is just the definition of Δg between the two given points and the second follows since the middle two terms cancel.)

Our *predictor* tool tells us that

$$g(p + \Delta x, q) - g(p, q) \sim \Delta x \frac{\partial g}{\partial x}(p, q).$$

(You can see this by dividing both sides by Δx. Treat $\sim$ as you would $=$. This is just the definition of the partial derivative of g with respect to x at (p, q).)

Similarly,

$$g(p + \Delta x, q + \Delta y) - g(p + \Delta x, q) \sim \Delta y \frac{\partial g}{\partial y}(p + \Delta x).$$

Observe that $\Delta g = 0$ since both $g(p, q) = c$ and $g(p+\Delta x, q+\Delta y) = c$. After substituting we have

$$0 \sim \Delta x \frac{\partial g}{\partial x}(p, q) + \Delta y \frac{\partial g}{\partial y}(p + \Delta x, q).$$

Hence

$$\Delta x \frac{\partial g}{\partial x}(p, q) \sim -\Delta y \frac{\partial g}{\partial y}(p + \Delta x, q)$$

and

$$\frac{\Delta y}{\Delta x} \sim -\frac{\frac{\partial g}{\partial x}(p, q)}{\frac{\partial g}{\partial y}(p + \Delta x, q)}.$$

Now let $\Delta x \to 0$. We conclude

$$\frac{dy}{dx} = -\frac{\frac{\partial g}{\partial x}(p, q)}{\frac{\partial g}{\partial y}(p, q)}.$$