

Midterm Solutions

Derivatives

1. For each function, decide whether the graph has at least one local max, at least one local min, or neither. (A point (p, q) is a *local max* for f if $f(x, y) \leq f(p, q)$ for all (x, y) sufficiently near (p, q) .)

(a) $f(x, y) = x^2 - y^2$ Circle one: Local max Local Min Neither

Solution: Simply look at the graph which is a saddle to see there are neither local minima nor local maxima. You may also use the second derivative test.

(b) $f(x, y) = 3x^2 + 2y^2$ Local max Local Min Neither *Solution:*

The graph is an elliptical cup and so has a local minimum. You may also apply the second derivative test.

(c) $f(x, y) = -x - y$ Local max Local Min Neither *Solution:*

Neither. This is a plane that has no (local) maximum or minimum points.

(d) $f(x, y) = (x - 1)^2$ Local max Local Min Neither

Solution: The graph is a trough. Every point along the bottom of the trough of the form $(1, y)$ is a local minimum.

2. True-False

(a) The function $f(x, y) = x + y$ has a local minimum constrained to the curve $y = x^2$. *T* *F*

Solution: True. The graph of f is a plane that slants upwards along the x - and y -axes. The constraint curve is a parabola which has a local minimum at the origin. You can also use Lagrange Multipliers.

(b) There exists a smooth function $f(x, y)$ which has a level curve consisting of two concentric circles. *T* *F*

Solution: True, consider a volcano shaped surface.

- (c) The mixed partials $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are always the same, no matter what the function f . T F

Solution: True. (We assume the function is smooth, so all the derivatives exist.)

- (d) If A is a level curve of a function $f(x, y)$ then A is the graph of another function $g(x)$. T F

Solution: False. Level curves of $f(x, y) = x^2 + y^2$ are circles and thus are not functions of a single variable.

Integration

Your answers to the following two questions should be a real number and may involve rationals such as Π and e

3. The density of an ant hill of radius 10 inches is given by $f(r) = 25e^{-r^2}$ in hundreds of ants per square inch. What is the total population of the ant hill? .

Solution. For radius 10:

$$\begin{aligned} & \int_0^{10} 2\pi r f(r) dr \\ &= \int_0^{10} 2\pi r \cdot 25e^{-r^2} dr \\ &= -25\pi(e^{-r^2})|_0^{10} \\ &= -25(e^{-100} - 1) \\ &= 25(1 - e^{-100}) \end{aligned}$$

For radius 25:

$$\begin{aligned} & \int_0^{25} 2\pi r f(r) dr \\ &= \int_0^{25} 2\pi r \cdot 10e^{-r^2} dr \\ &= -10\pi(e^{-r^2})|_0^{25} \\ &= -10(e^{-625} - 1) \\ &= 10(1 - e^{-625}) \end{aligned}$$

4. Use a double integral to find the integral of the function $f(x, y) = 1$ over the domain D which is defined to be the region between the two curves $h(x) = 1 + \sin x$ and $g(x) = 1 + x$ with $0 \leq x \leq 1$.

For $h(x) = 1 + \sin x, g(x) = 1 + x$:

Notice $1 + x \geq 1 + \sin x$ for all $x \geq 0$, so

$$\begin{aligned}
 \int \int_D f(x, y) \, dx \, dy &= \int_0^1 \left(\int_{h(x)}^{g(x)} f(x, y) \, dy \right) dx \\
 &= \int_0^1 \left(\int_{h(x)}^{g(x)} 1 \, dy \right) dx \\
 &= \int_0^1 (g(x) - h(x)) \, dx \\
 &= \int_0^1 (x - \sin x) \, dx \\
 &= \left(\frac{1}{2}x^2 + \cos x \right) \Big|_0^1 \\
 &= \left(\frac{1}{2} + \cos(1) \right) - (0 + 1) \\
 &= \cos(1) - \frac{1}{2}
 \end{aligned}$$

For $h(x) = 2 + \cos x$, $g(x) = 1 + x$:

Notice $2 + x \geq 1 + \cos x$ for all $x \geq 0$, so

$$\begin{aligned}
 \int \int_D f(x, y) \, dx \, dy &= \int_0^1 \left(\int_{h(x)}^{g(x)} f(x, y) \, dy \right) dx \\
 &= \int_0^1 \left(\int_{h(x)}^{g(x)} 1 \, dy \right) dx \\
 &= \int_0^1 (g(x) - h(x)) \, dx \\
 &= \int_0^1 (1 + x - \cos x) \, dx \\
 &= \left(x + \frac{1}{2}x^2 - \sin x \right) \Big|_0^1 \\
 &= \left(1 + \frac{1}{2} - \sin(1) \right) - (0 + 0 - 0) \\
 &= \frac{3}{2} - \sin(1)
 \end{aligned}$$

Differential equations

5. Suppose that a deposit is made to a bank account paying an annual interest rate of 5% compounded continuously. No other deposits or withdrawals are made to the account.

- (a) Write a differential equation satisfied by $B(t)$, the balance in the account after t years.

Solution. $\frac{dB}{dt} = 0.05B(t)$.

- (b) What is the general solution to this differential equation?

Solution. Solve by separating variables, to obtain

$$\int \frac{1}{B(t)} dB = \int 0.05 dt.$$

Then

$$\log B(t) = 0.05t + c$$

for some constant c , and exponentiating we obtain

$$B(t) = Ae^{0.05t}$$

for some constant A .

- (c) If the initial deposit is \$4000, give the particular solution satisfying this initial condition.

Solution. At $t = 0$, we have $B(0) = Ae^0 = A$, and so we must have $A = 4000$. Thus the particular solution is

$$B(t) = 4000e^{0.05t}.$$

- (d) How much is in the account after 30 years? (Write your answer in the simplest terms you can without a calculator.)

Solution. We now evaluate at $t = 30$, to obtain

$$B(30) = 4000e^{(0.05)(30)} = 4000e^{1.5}.$$