MATH 16B: ANALYTIC GEOMETRY AND CALCULUS
QUIZ #3 SOLUTIONS

Question 1. *Calculate the integral of* $\cos(\frac{y}{x})$ *over the region bounded by the lines* $x = 0$, $x = a$, $y = 0$ *and* $y = \frac{\pi x}{2}$.

Solution. We need to calculate the integral

$$\begin{aligned}
\int_0^a \int_0^{\frac{\pi x}{2}} \cos\left(\frac{y}{x}\right)\ \mathrm{d}y\mathrm{d}x &= \int_0^a x\sin\left(\frac{y}{x}\right)\Big|_0^{\frac{\pi x}{2}}\ \mathrm{d}x \\
&= \int_0^a x\sin\left(\frac{\pi}{2}\right) - x\sin(0)\ \mathrm{d}x \\
&= \int_0^a x\ \mathrm{d}x \\
&= \frac{x}{2}\Big|_0^a \\
&= \frac{a^2}{2}
\end{aligned}$$

Question 2. *When I throw darts at a dartboard, they're randomly distributed around the centre, according to a normal distribution. This means that the probability a dart lands within a disc of radius R feet about the bullseye is calculated by integrating $\frac{1}{\pi}e^{-r^2}$ over that disc.*

If the dartboard has radius R feet, write down the integral which calculates the probability that I hit it at all.

Use the following indefinite integral (or use integration by substitution, if you know that), to calculate the above integral.

$$\int re^{-r^2}\ dr = -\frac{1}{2}e^{-r^2}$$

(Hint — you can see if you've got the right answer by checking that for really large dartboards, I nearly always hit it.)

(Bonus question — for a regulation 18-inch dartboard, do I hit it more often than I miss it?)

Solution. Lots of people had trouble interpreting this question, so it was decided to make it a bonus question.

The key idea was that we have a function $\frac{1}{\pi}e^{-r^2}$ defined on a circular region, where r is the distance to the centre. You should think of 'integrating this function over the disc' as calculating the volume above the disc, and below the surface defined by this function. We've learnt that integrals over discs are calculated by integrating $2\pi r$ times the function over the interval from 0 to the radius of the disc.

That is,

$$\begin{aligned}\int_0^R 2\pi r \frac{1}{\pi} e^{-r^2} dr &= 2\int_0^R re^{-r^2} dr \\ &= 2\left(-\frac{1}{2}\right) e^{-r^2}\Big|_0^R \\ &= -e^{-R^2} + e^0 \\ &= 1 - e^{-R^2}.\end{aligned}$$

If you've seen something about probability, you'll understand that this means that for really large dartboards, I nearly always hit, because $1 - e^{-R^2}$ approaches 1 as R goes to infinity.

For an 18-inch dartboard, $R = 1.5$, and so $-R^2 = -2.25 < -2$, so $e^{-R^2} < e^{-2} < 2^{-2} = \frac{1}{4}$, so $1 - e^{-R^2} > \frac{3}{4}$. Thus at least three quarters of the time (in fact even more) I hit the dartboard — and so hit more often than I miss!